USE OF THE METHOD OF SUCCESSIVE INTERVALS TO SOLVE NONLINEAR PROBLEMS OF HEAT CONDUCTION

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The possibility of solving a nonlinear problem of heat conduction by the method of successive intervals is considered. A description is given of the algorithm realizing this method. The effectiveness of applying the method is discussed and a comparison with the mesh method is presented.

A number of questions associated with heat transfer in solids whose thermophysical parameters depend strongly on the temperature (as an example, the coefficient of heat conduction of silicon varies 170- and of germanium 60-fold in the 0-1000°K range) arises in the investigation of the heat modes of radioelectronic apparatus (REA). The necessity to compute the temperature field of semiconductor instruments requires the solution of a nonlinear problem of heat conduction. Numerical methods [1] are ordinarily used to solve such problems.

Let us examine the possibility of using the method of successive intervals [2] to solve the nonlinear problem.

It is known that the solution of an incorrect problem about the temperature field in the following formulation:

$$\frac{\partial t(x,\ \tau)}{\partial \tau}=a\frac{\partial^2 t(x,\ \tau)}{\partial x^2} \tag{1}$$

for $0 \le \tau \le \infty$ and $0 \le x \le R$

$$t(x,\ 0)=0, \tag{2}$$

$$-\lambda\frac{\partial t(x,\ \tau)}{\partial x}\bigg|_{x=R}=q_1(\tau), \tag{3}$$

$$t(R,\ \tau)=t_1(\tau), \tag{4}$$

when $q_2(\tau)$ and $t(x,\ \tau)$ must be found, results in a Volterra integral equation of the first kind:

$$\int_0^\tau t(x,\ \eta)\frac{\exp\left[-\frac{(R-x)^2}{4a(\tau-\eta)}\right]}{(\tau-\eta)^{3/2}}d\eta=\frac{\sqrt{\pi a}}{R-x}t(R,\ \tau)+$$
$$+\int_0^\tau t(R,\ \eta)\frac{\exp\left[-\frac{(R-x)^2}{a(\tau-\eta)}\right]}{(\tau-\eta)^{3/2}}d\eta+\frac{a}{\lambda(R-x)}\int_0^\tau q_1(\eta)\times$$
$$\times\frac{d\eta}{(\tau-\eta)^{1/2}}-\frac{a}{\lambda(R-x)}\int_0^\tau q_1(\eta)\frac{\exp\left[-\frac{(R-x)^2}{a(\tau-\eta)}\right]}{(\tau-\eta)^{1/2}}d\eta. \tag{5}$$

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Some numerical method is also used ordinarily to determine the temperature field by means of this solution.

But the solution of the problem (1)-(4) can be obtained by using the solution in successive intervals for the temperature field in the wall during a process of nonstationary heat transmission through it. For a zero initial temperature distribution, this solution has the following form [3]:

$$t(y,\ n\omega) = \frac{R}{\lambda}\left\{\sum_{i=1}^{n} \bar{q}_{1,i}\,\Delta F(y,\ z_i\omega) - \sum_{i=1}^{n} \bar{q}_{2,i}\Delta F(1-y,\ z_i\omega)\right\}. \tag{6}$$

Hence, the magnitude of the desired heat flux $q_2(\tau)$ will be determined as follows in the first interval:

$$q_{2,1} = \frac{1}{\Delta F(y,\ \omega)}\left\{q_{1,1}\Delta F\,[(1-y_1),\ \omega] - \frac{\lambda}{R}\,t(y_1,\ \omega)\right\}, \tag{7}$$

and by the expression

$$q_{2,n} = \frac{1}{\Delta F(y,\ \omega)}\left[\left\{\sum_{k=0}^{n-1} q_{1,k+1}\Delta F\,[(1-y_1);\ (n-k)\,\omega] - \sum_{n=0}^{n-2} q_{2,k+1}\Delta F\,[y_1,\ (n-k)\,\omega]\right\} - \frac{\lambda}{R}\,t(y,\ n\omega)\right] \tag{8}$$

in all the following intervals for $2 \le n \le N$. Here it is assumed that $y = x/R$ and $\omega = \Delta Fo$. Having determined the whole sequence of values $q_{2,k}$, or, in other words, having determined the function $q_2(\tau)$, the temperature field of a plate, including the temperature at $x = 0$ also, can be determined by means of the solution (6). Therefore, the solution of the nonsymmetric problem of heat conduction in successive intervals permits computation of the temperature field when the heat-exchange conditions are known only on one body surface (plate) in the form (3)-(4).

This singularity of the solution in successive intervals can be used to obtain an approximate solution of the nonlinear problem of heat conduction which can be formulated as follows:

$$c(t)\,\rho(t)\,\frac{\partial t(x,\ \tau)}{\partial\tau} = \frac{\partial}{\partial x}\left[\lambda(t)\,\frac{\partial t(x,\ \tau)}{\partial x}\right] \tag{9}$$

for $0 \le \tau \le \infty$ and $0 \le x \le R$

$$t(x,\ 0) = 0, \tag{10}$$

$$-\lambda(t)\,\frac{\partial t(x,\ \tau)}{\partial x}\bigg|_{x=R} = q_1(\tau), \tag{11}$$

$$t(R,\ \tau) = t_1(\tau). \tag{12}$$

It is required to determine the temperature field in a plate, including $t(0,\ \tau)$ and the heat flux $q_2(\tau)$ passing through the surface $x = 0$.

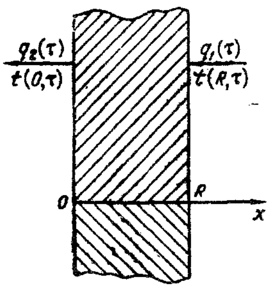
To obtain the desired algorithm of the solution of the nonlinear problem (9)-(12) let us partition the body (plate) into a number of domains (Figs. 1 and 2) and starting from the given temperature on the boundary of the first domain let us select the value of the thermophysical parameters in this domain. Then, using the solution of the incorrect problem (1)-(4) in successive intervals, let us determine the temperature and heat flux on the second boundary of the spatial domain in the given time interval. Furthermore, let us find the temperature and heat flux on the boundary of this same spatial domain for the next time interval. Performing analogous calculations n times, we obtain the appropriate boundary conditions for the second spatial domain. The thermophysical characteristics in this domain are selected on the basis of the temperature obtained on the boundary of the second spatial domain. Repeating this iteration process over the whole set of spatial domains, we obtain the solution of the nonlinear problem of heat conduction. The computational relationships obtained by using the method considered above are the following:

$$q_{i+1,1} = \frac{1}{\Delta F(y_i,\ \omega)}\left\{q_{i,1}\Delta F\,[(1-y_i),\ \omega] - \frac{\lambda_{i,1}}{R}\,t(y_i,\ \omega)\right\} \tag{13}$$

for $n = 1$ and $1 \le i \le I$

$$q_{i+1,n} = \frac{1}{\Delta F(y_i,\ \omega)}\left[\left\{\sum_{k=0}^{n-1} q_{i,k+1}\Delta F\,[1-y_i,\ (n-k)\,\omega] - \sum_{k=0}^{n-2} q_{i+1,k+1}\Delta F\,[y_i,\ (n-k)\,\omega]\right\} - \frac{\lambda_{i,n}}{R}\,(y_i,\ n\omega)\right] \tag{14}$$

for $2 \le n \le N$ and $1 \le i \le I$



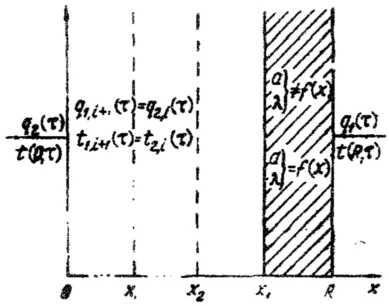


Fig. 1 Fig. 2

Fig. 1. Infinite plate with nonsymmetric boundary conditions.

Fig. 2. Partition scheme for the domain.

$$t(y_{i+1},\ n\omega) = \frac{R}{\lambda_{i,n}}\left\{\sum_{k=0}^{n-1} q_{i,k+1}\Delta F\,[1-y_{i+1},\ (n-k)\,\omega] - \sum_{k=0}^{n-1} q_{i+1,k+1}\Delta F\,[y_{i+1},\ (n-k)\,\omega]\right\} \tag{15}$$

for $1 \le n \le N$ and $1 \le i \le I$. Since the process of calculation using the formulas presented above is an iteration, then the question of the stability of the method used arises, which reduces to the question of the stability of the inverse problem in successive intervals with boundary conditions of the second kind. It is known [4] that the magnitude of the selected time interval influences the stability of such a solution and a limit $\omega_{min} = z^2/2$ exists (z is the relative distance from the heat-exchange surface) below which the stability of the solution is spoiled. In this case $\omega_{min} \simeq 0$, i.e., the solution of the system (9)-(12) in successive intervals is absolutelv stable.

To verify the effectiveness of this approach to the solutions of nonlinear problems from the viewpoint of accuracy of the calculations and machine time expenditure, a problem in the incorrect formulation (1)-(4) was computed by the mesh method and by the method of successive intervals. The solution of the linear problem of heat conduction, when the boundary conditions are given in the following formulation:

$$\frac{\partial t(R,\ \tau)}{\partial x} = -\frac{1}{\lambda}\, q_0\,[1+\exp(-b\tau)],$$

$$\frac{\partial t(0,\ \tau)}{\partial x} = -\frac{1}{\lambda}\, q_0\,[1-\exp(-b\tau)]$$

was hence used as the standard solution. The computation was carried out for a body with the characteristics R = 0.05 m; λ = 69.78 W/(m·deg); $a = 0.1667 \cdot 10^{-4}$ m²/sec; q_0 = 17400 W/m², b = 0.007 sec⁻¹.

The computations carried out showed that for a ω = 0.1 duration of the time interval and partition of the body into four spatial domains, the error in reproducing the desired boundary conditions after the seventh time interval is tenths and hundredths of a percent. While errors on the order of 1% in

TABLE 1. Computation of the Boundary Conditions for x = 0

Fo=nω	Mesh method (z = 10)				Method of successive intervals (z = 4)			
	$q_2(\tau)$	δq_2, %	$t(0,\tau)$	δt, %	$q_2(\tau)$	δq_2, %	$t(0,\tau)$	δt, %
0,1	92	88,0	—1,91	—1797,5	—32894	4387,7	9,53	9572,5
0,2	3936	—38,6	—0,72	205,5	54123	—2376,1	—8,41	1329,0
0,3	5910	—45,8	0,83	58,5	—21424	718,6	5,84	—192,6
0,4	7140	—38,8	2	29,8	13401	—190,5	2,26	33,7
0,5	8045	—31,3	3,87	18,9	3542	37,3	5,07	—6,3
0,6	8783	—25,2	5,22	13,4	7034	—6,9	5,99	0,5
0,7	9420	—20,7	6,44	10,1	7286	1,8	7,20	—0,5
0,8	9982	—17,1	7,55	7,9	8144	0,4	8,22	—0,3
0,9	10485	—14,3	8,55	6,4	8806	0,6	9,15	—0,2
1,0	10935	—12,2	9,44	5,3	9425	0,5	9,99	—0,2
1,5	12596	— 5,9	12,76	2,4	11701	0,2	13,08	—0,1
2,0	13578	— 3,2	14,72	1,2	13048	0,1	14,91	—0,0
2,5	14158	— 1,8	15,88	0,7	13845	0,1	15,99	—0,0
3,0	14447	— 1,1	16,45	0,4	14317	0,0	16,63	—0,0

TABLE 2. Computation of the Boundary Conditions at $x = 0$

Fo	Mesh method for z				Method of successive intervals ($z = 10$)
Relative error in determining the heat-flux density $\Delta q/q$ in %					
	$z=10$	$z=20$	$z=50$	$z=100$	
0,5	—	—	—	—	60,3
1,0	—73,3	—64,0	—58,6	—56,8	—29,3
1,5	—27,6	—24,2	—22,1	—21,5	5,6
2,0	—13,4	—11,7	—10,7	—10,4	1,0
2,5	— 7,2	— 6,3	— 5,7	— 5,6	0,6
3,0	— 4,0	— 3,5	— 3,2	— 3,1	0,4
Relative error in determining the surface temperature $t(0, \tau)$ in %					
0,5	—	—	—	—	—50,3
1,0	42,8	37,9	35,0	34,1	4,8
1,5	11,9	10,5	9,7	9,4	— 1,3
2,0	5,3	4,7	4,4	4,2	— 0,4
3,0	1,5	1,4	1,2	1,2	— 0,2

reproducing the desired boundary conditions are reached in the mesh method for a 10×10 space–time mesh only after 20–25 time intervals (see Table 1), the machine times expended in computations by both these methods were equal.

However, for a $\omega = 0.5$ duration of the time interval, the accuracy of the computations by the successive interval method for a partition into 10 spatial domains is greater than the accuracy of the mesh method, even with a partition into 100 spatial domains (see Table 2), which is equivalent to a fourfold gain in machine time.

The higher accuracy in reproducing the desired boundary conditions in the successive intervals method as compared with the mesh method is evidently related to the fact that the initial differential equation in the mesh method is replaced by finite differences, while a stepwise approximation of the functions describing the time change in the boundary conditions in a given spatial domain is made in the successive-intervals method, hence the form of the initial differential equation remains unchanged for this domain.

The use of the proposed approach to solve problems with sources and to process experimental data for an essential temperature dependence of the physical parameters of the solid is the subject of a separate paper.

NOTATION

t, temperature; x, coordinate; τ, time; a, thermal diffusivity; λ, coefficient of thermal conductivity; $c\rho$, volumetric specific heat; q_i, heat-flux density; η, variable of integration; $\Delta F(y, n\omega)$, reference function from [3].

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